Math 5B Final Exam – K. Hogue
Spring 2021 v1
Instructions on Canvas **200 POINTS**

FILL IN THE BLANK WITH THE MOST APPROPRIATE ANSWER. NO PARTIAL CREDIT. (4 POINTS EACH)

(1) TRUE OR FALSE: If $0 \le a_n \le b_n$ and $\sum a_n$ diverges, then $\sum b_n$ diverges True

(2) Express the point $\left(-\sqrt{3},1\right)$ in polar coordinates(exactly) $(2, \frac{5\pi}{6})$

(3) $\int \frac{1}{1+x^2}\,dx =$ $\tan^{-1}x + C$

(4) $\frac{d}{dx}\sin^{-1}(3x) =$ $\frac{3}{\sqrt{1-9x^2}}$

(5) $\frac{d}{dx}\left(\frac{x^3}{\ln(5x)}\right) =$ $\frac{3x^2\ln(5x) - x^2}{(\ln(5x))^2}$ (simplify)

(6) $\int e^{4x}\,dx =$ $\frac{1}{4}e^{4x} + C$

(7) Given $\sum_{n=1}^{\infty} a_n$, if $\lim_{n\to\infty} a_n \ne 0$ what is known about the convergence/divergence of the series? diverges

(8) If $f(x) = 5 - \ln x$, find $f^{-1}(x)$ $= e^{5-x}$

$y = 5 - \ln x$
$x = 5 - \ln y$
$\ln y = 5 - x$

(9) $\lim_{x\to 0^+} x^2 \ln x =$ 0 (show work clearly)

$= \lim_{x\to 0^+} \frac{\ln x}{1/x^2} \overset{L}{=} \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+} -\frac{x^2}{2} = 0$

(10) Find the sum exactly: $\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n}$ 1

geometric $r = \frac{2}{3} < 1 \Rightarrow$ converges. $a = \frac{1}{3}$

$S = \frac{a}{1-r} = \frac{1/3}{1-2/3}$

(11) Solve the differential equation $y' = x^4 y$ with initial condition $y(0) = 3$. Solve for y explicitly. Be sure to show constants carefully. **(11 points)**

$$\frac{dy}{dx} = x^4 y$$

$$\frac{dy}{y} = x^4 dx$$

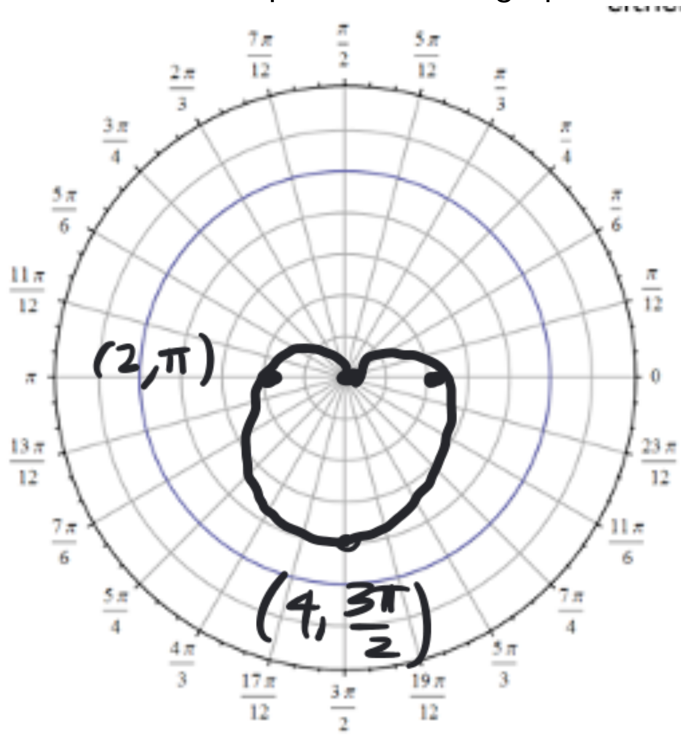
$$\ln|y| = \frac{1}{5}x^5 + c_1$$

$$y = e^{\frac{1}{5}x^5 + c_1} = e^{1/5 x^5} e^{c_1}$$

$$e^{c_1} \to C$$

$$\boxed{y = Ce^{\frac{1}{5}x^5}}$$

(12) (a). Graph the polar curve r= 2-2sinθ Label two *polar* points ON the graph
(b). Find the area of the portion of the graph in the first quadrant. **(22 points)**



a) This is a cardioid.- plot quadrantal points

b) $A = \frac{1}{2}\int r^2 d\theta = \frac{1}{2}\int_0^{\pi/2} (2-2\sin\theta)^2 d\theta$

$= \frac{1}{2}\int_0^{\pi/2} (4 - 8\sin\theta + 4\sin^2\theta) d\theta$

$= \frac{1}{2}\left(4\theta + 8\cos\theta + 4\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\right)\Big|_0^{\pi/2}$

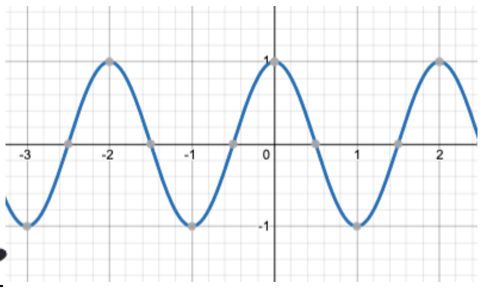
$= \frac{1}{2}\left(2\pi + 4\left(\frac{\pi}{4}\right) - (8)\right) = \boxed{\frac{1}{2}(3\pi - 8)}$

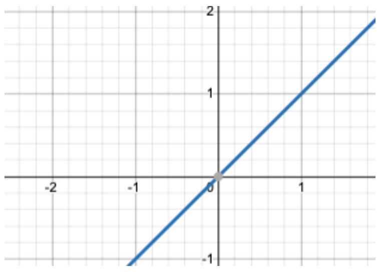
(13) (12 points)

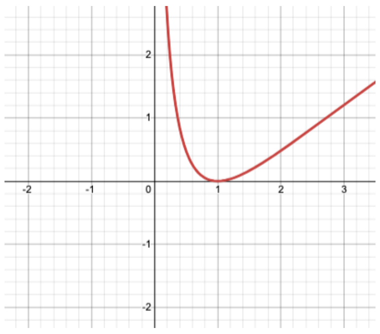
Match the graphs of the parametric pair x(t) and y(t) on the left with the graph in the xy plane on the right.

x(t) y(t)

(a) C

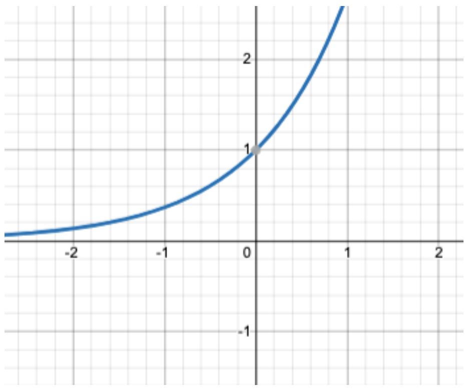


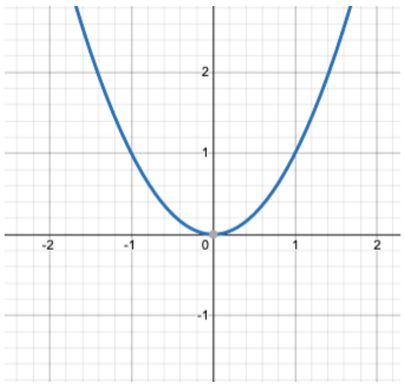


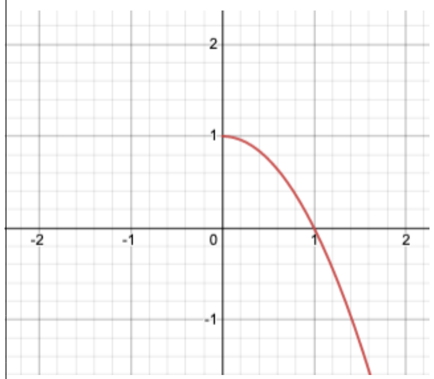


A

(b) A

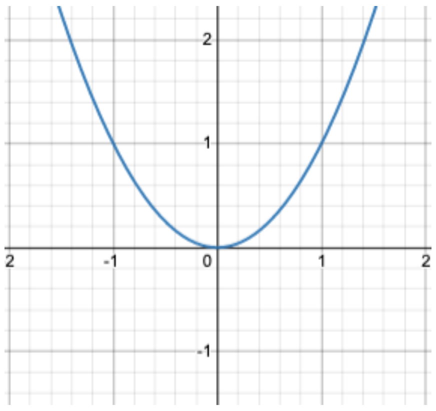


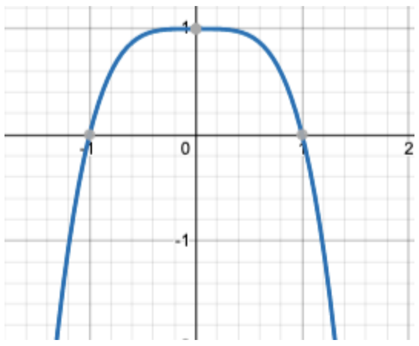


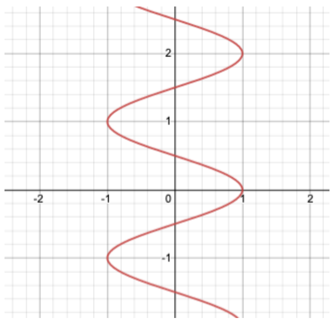


B

(c) B

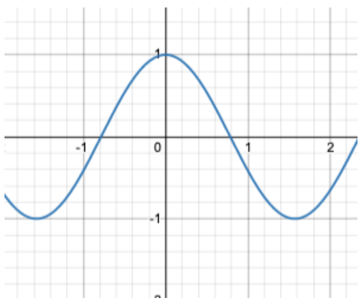


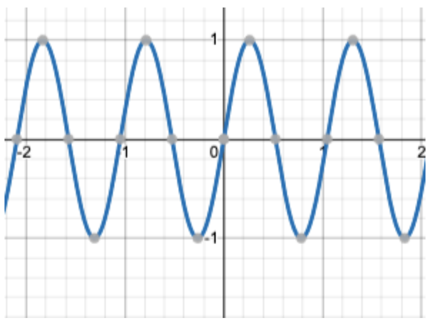


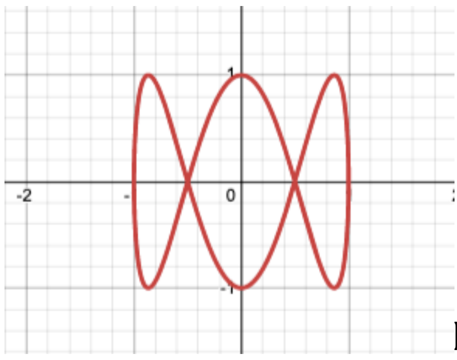


C

(d) D







D

c

(14) For each of the following series, classify as convergent (absolute or conditional if applicable) or divergent. SHOW ALL DETAILS. (15 points each)

(a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

Absolute Convergence?

$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$

$\frac{1}{\ln n} > 0$ and $\frac{1}{\ln n} > \frac{1}{n}$.

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (harmonic)

Then $\sum \frac{1}{\ln n}$ diverges by comparison test so

$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is not Abs. Conv.

Apply AST

$b_n = \frac{1}{\ln n}$

i) clearly $\frac{1}{\ln(n+1)} < \frac{1}{\ln n}$

and ii) $\lim_{n\to\infty} \frac{1}{\ln n} = 0$

So by AST, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is convergent

So given series is conditionally conv.

(b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{5^n}{(3n)!}$

Ratio Test

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{5^{n+1}}{3(n+1)!} \cdot \frac{(3n)!}{5^n} \right|$$

$$= \frac{5\,(3n)!}{(3n+3)!}$$

$$= \frac{5}{(3n+3)(3n+2)(3n+1)}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$$

So given series is absolutely convergent

(15) Clearly show the integral test applies to the following series and use it to determine whether the series converges or diverges. Correct mathematical notation is expected.

(20 points)

$$\sum_{n=1}^{\infty} n^2 e^{-n}$$

$f(x) = x^2 e^{-x}$

i) $f > 0$ for $x > 0$

ii) f cont's for all x

iii) f decreasing for $x \geq 2$, since:

$$f'(x) = 2xe^{-x} - x^2 e^{-x} = e^{-x} x (2-x)$$

$- \quad + \quad -$

$0 \quad 2$

So integral test applies

Consider indefinite integral

$$\int x^2 e^{-x}\,dx$$

By parts twice

$u = x^2 \qquad dv = e^{-x}\,dx$

$du = 2x\,dx \qquad v = -e^{-x}$

$$= -x^2 e^{-x} + 2\int x e^{-x}\,dx$$

$u = x \qquad dv = e^{-x}\,dx$

$du = dx \qquad v = -e^{-x}$

$$= -x^2 e^{-x} + 2\left(-xe^{-x} + \int e^{-x}\right)$$

$$= -x^2 e^{-x} - 2xe^{-x} + 2e^{-x} + C$$

So $\displaystyle\int_1^{\infty} x^2 e^{-x}\,dx = \lim_{t\to\infty} \int_1^t (x^2 e^{-x})\,dx = \lim_{t\to\infty} \left(-x^2 e^{-x} - 2xe^{-x} + 2e^{-x}\right)\Big|_1^t$

$$= \lim_{t\to\infty} -t^2 e^{-t} - \lim_{t\to\infty} 2te^{-t} + \lim_{t\to\infty} e^{-t} - (2) = -2$$

each of these $\to 0$ but you need to show it. (L'Hop on first 2)

$\Rightarrow$ integral converges so series converges.

(16) Compute each of the following integrals:

(a) $\int_0^1 \frac{x^3}{\sqrt{4-x^2}}\,dx$ You must use trigonometric substitution on this one. No credit for a different method

(20 points)

let $x = 2\sin\theta \Rightarrow \sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$

$dx = 2\cos\theta\,d\theta$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\int_0^1 \frac{x^3}{\sqrt{4-x^2}}\,dx = \int_0^{\pi/6} \frac{8\sin^3\theta}{2\cos\theta} 2\cos\theta\,d\theta = 8\int_0^{\pi/6} \sin^3\theta\,d\theta$$

$$= 8\int_0^{\pi/6} \sin\theta(1-\cos^2\theta)\,d\theta \qquad u = \cos\theta \quad du = -\sin\theta\,d\theta$$

$$= 8\int (u^2-1)\,du = 8\left(\frac{1}{3}u^3 - u\right)\Big] =$$

$$= \frac{8}{3}\cos^3\theta - 8\cos\theta\Big]_0^{\frac{\pi}{6}} = \frac{8}{3}\left(\frac{\sqrt{3}}{2}\right)^3 - 8\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{8}{3} - 8\right)$$

$$= \sqrt{3} - 4\sqrt{3} + \frac{16}{3} = \boxed{\frac{16}{3} - 3\sqrt{3}}$$

(b) $\int_e^5 \frac{1}{x(\ln x)^2}\,dx$ $\qquad u = \ln x \quad du = \frac{1}{x}dx$

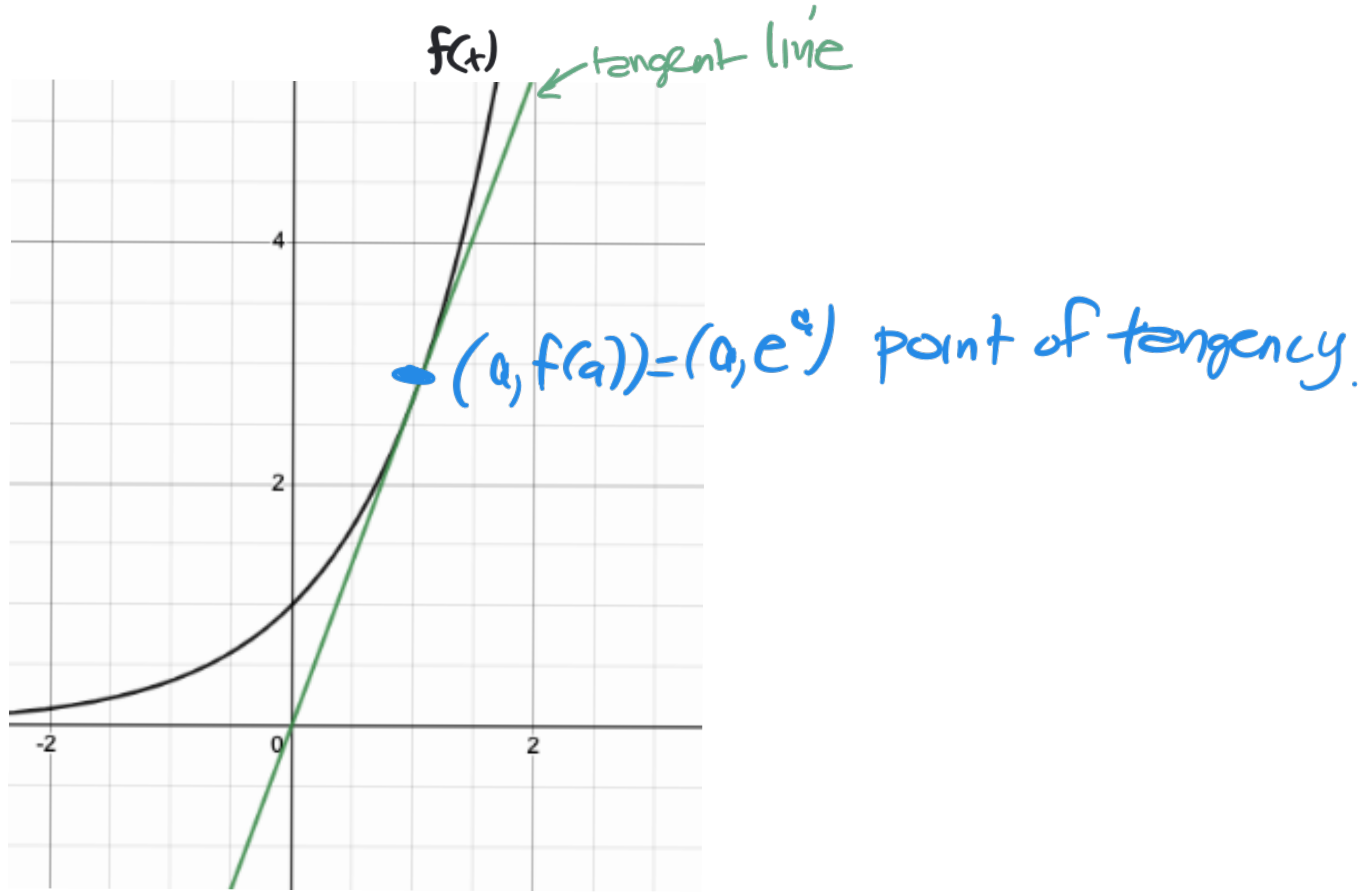
(10 points)

$$\int_1^{\ln 5} \frac{1}{u^2}\,du = -u^{-1}\Big]_1^{\ln 5} = -(\ln 5)^{-1} + 1$$

$$= \boxed{1 - \frac{1}{\ln 5}}$$

(17) (a). Find the equation of a tangent line to $f(x) = e^x$ which contains the origin (10 points)

(b) Sketch $f(x)$ and the tangent line from part (a).



First, find general form of line tangent to f(x) at $(a, f(a)) = (a, e^a)$.

$$f'(x) = e^x$$

$$m = e^a$$

line: $y - e^a = e^a(x - a)$

Must pass through $(0,0) \Rightarrow$

$$0 - e^a = e^a(0 - a)$$

$$a = 1$$

$$\boxed{y - e = e(x-1)}$$

(18) (a) Use series to approximate $\int_0^{1/2} x\cos(x^2)\,dx$ with error less than 0.00001 .(15 points)

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2}$$

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} = 1 - \frac{x^4}{2}$$

$$x\cos x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!} = x - \frac{x^5}{2}$$

$$\int_0^{1/2} x\cos(x^2)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n)!(4n+2)}\Bigg]_0^{1/2} \qquad \frac{x^2}{2} - \frac{x^6}{12}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (1/2)^{4n+2}}{(2n)!(4n+2)} \qquad \frac{1}{8} - \frac{1}{64 \cdot 12}$$

$$= \left[\boxed{\frac{1}{8} - \frac{1}{2^6 \cdot 2 \cdot 6}}\right] + \frac{1}{2^{10}\, 4! \cdot 10} - \cdots$$

$.0007$

$\approx \boxed{.1236979}$

$.0000004$

first term $< .00001$

(b) Find the value of the integral exactly by integrating directly. (10 points)

$u = x^2$

$du = 2x\,dx$

$$\int_0^{1/2} x\cos(x^2)\,dx$$

$$= \frac{1}{2}\int_0 \cos u\,du$$

$$= \frac{1}{2}\sin(x^2)\Big]_0^{1/2}$$

$$= \boxed{\frac{1}{2}\sin\left(\frac{1}{4}\right)} \approx .12370197$$

note: these should be within .00001 of each other.